

EXACT DISTRIBUTION OF INTERMITTENTLY CHANGING POSITIVE AND NEGATIVE COMPOUND POISSON PROCESS DRIVEN BY AN ALTERNATING RENEWAL PROCESS AND RELATED FUNCTIONS

Short title for running head: Up and Down CPP

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Abstract Alternating renewal processes have been widely used to model social and scientific phenomena where independent “on” and “off” states alternate. In this article we study a model where the value of a process cumulates and declines according to two modes of compound Poisson processes (CPP) with respect to an underlying renewal process. The model discussed in the present paper can be used as a revenue management model applied to inventory or to finance. The exact distribution of the process is derived as well as the double Laplace transform with respect to the level and time of the process.

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1 Introduction

Brownian motions whose trends follow a generalized telegrapher process were studied by Di Crescenzo and Zacks [5] and Di Crescenzo et al. [4]. Applications of such processes were mentioned in finance, physics, and other areas. In the present paper, we study up and down compound Poisson processes which are driven by generalized telegrapher processes. The reader is referred to Zacks [6] for the properties of a generalized telegrapher process.

To strengthen the motivation, let us introduce our model as a revenue management model applied to inventory. That is, there are two periods: stocking period and selling period that alternate according to two modes of compound Poisson processes (CPP). During the stocking period the purchase price is low so that one only buys and during the selling period the price is high so that one only sells. Another example of our model would be a storage model where goods arrive or leave a warehouse according to two alternating phases. In cases where arrival and departure processes consist of sequences of demands of independent identically distributed amounts where each demand occurs on Poisson clock, it is appropriate to model the arriving and departure processes by compound Poisson processes (CPP). The underlying alternating phases can be thought as an alternating renewal process. Motivated by this model we study a general “up and down” stochastic process $\{Y(t), t \geq 0\}$ which, for a random amount of time U , follows a compound Poisson process $\{Y_1(t), t \geq 0\}$ and then for a random amount of time V , follows a negative compound Poisson process $\{-Y_2(t), t \geq 0\}$, intermittently. Formally, $\{Y_i(t), t \geq 0\}$, $i = 1, 2$ are defined as

$$Y_i(t) = \sum_{n=0}^{N_i(t)} X_n^{(i)}, \quad i = 1, 2, \quad (1.1)$$

where $\{N_i(t), t \geq 0\}$ is a Poisson counting process with intensity $\lambda_i > 0$, and $\{X_n^{(i)}, n \geq 1\}$ are i.i.d. positive random variables having an absolutely continuous distribution F_i , with density f_i . We also set $X_0^{(i)} \equiv 0$. $\{X_n^{(i)}, n \geq 1\}$ and $\{N_i(t), t \geq 0\}$ are mutually independent. In addition, let $\{U_1, V_1, U_2, V_2, \dots\}$ be an alternating renewal process. U_i and V_i ($i \geq 1$) are absolutely continuous positive random variables, having distribution functions F_U and F_V , with density functions f_U and f_V respectively. Moreover, let $\tau_0 = 0$, and

$$\tau_n = \sum_{i=1}^n (U_i + V_i), \quad n \geq 1, \quad (1.2)$$

be renewal epochs after the n -th renewal cycle. Each renewal cycle consists of one ‘‘up’’ period U_i followed by one ‘‘down’’ period V_i .

To put notations into perspective, goods (or data) arrive at a central storage (or process) unit during U_i and leave during V_i . The arrival and departure epochs follow two independent Poisson counting processes $N_1(t)$ and $N_2(t)$, and the amount of item (data) arriving or leaving at each epoch follows positive random variables $X_n^{(1)}$ and $X_n^{(2)}$ respectively (see Figure 1).

Let us denote $\mathbf{1}_{\{\cdot\}}$ as the indicator function. The process $Y(t)$ is defined formally as

$$Y(t) = \sum_{n=1}^{\infty} \left[\mathbf{1}_{\{\tau_{n-1} \leq t < \tau_{n-1} + U_n\}} (Y(\tau_{n-1}-) + Y_1(t - \tau_{n-1})) \right. \\ \left. + \mathbf{1}_{\{\tau_{n-1} + U_n \leq t < \tau_n\}} (Y((\tau_{n-1} + U_n)-) - Y_2(t - \tau_{n-1} - U_n)) \right], \quad (1.3)$$

for $t > 0$ and $Y(0) = 0$. Notice that $Y(0) = 0$ is not an essential assumption. The figure 1 illustrates a typical sample path of $Y(t)$.

Notice that the processes $Y_1(t)$ and $Y_2(t)$ might have no increase over some time period. Thus,

$$P\{Y_1(U_1) = 0\} = E\{e^{-\lambda_1 U_1}\} = M_U(-\lambda_1) \quad (1.4)$$

where M_U is the moment generating function of U . Similarly, $P\{Y_2(V_1) = 0\} = M_V(-\lambda_2)$.

The rest of the article is structured as follows. In Section 2, we derive the c.d.f of $Y(t)$ in

details in terms of the distributions of $Y_i(t), i = 1, 2$. In Section 3, we find the double Laplace transform of $Y(t)$. Finally, in Section 4, we present numerical results for the special case where $\{X_n^{(i)}, n \geq 1\}$ and $\{U_i\}, \{V_i\}$ are exponentially distributed.

2 Distribution of $Y(t)$

In this section, we derive the distribution function and the moments of $Y(t)$. Recall from Eq. (1.2) that τ_n denotes the time epoch at the end of the n -th renewal interval. Define the following random intervals

$$I_n^+ = [\tau_{n-1}, \tau_{n-1} + U_n), \quad (2.1)$$

and

$$I_n^- = [\tau_{n-1} + U_n, \tau_n), \quad (2.2)$$

On I_n^+ , $Y(t)$ is always non-decreasing and develops like a stochastic copy of $Y_1(t)$, while on I_n^- it is always non-increasing and develops like a stochastic copy of $-Y_2(t)$. Intervals I_n^+ will be referred to as “up” intervals, and I_n^- “as down” intervals. Let us define $W(t)$ to be the total “on” time before a fixed time t . Formally,

$$W(t) = \int_0^t \mathbf{1}_{\{s \in \cup_{n=1}^{\infty} I_n^+\}} ds. \quad (2.3)$$

Since CPP’s $Y_1(t)$ and $Y_2(t)$ are Lévy processes, i.e., processes with stationary independent increments, we have

$$Y(t) \sim Y_1(W(t)) - Y_2(t - W(t)). \quad (2.4)$$

Thus, the c.d.f. of $Y(t)$ at t can be written as

$$H_Y(y; t) := P\{Y(t) \leq y\} = P\{Y_1(W(t)) \leq y + Y_2(t - W(t))\}. \quad (2.5)$$

Zacks [7] derived the distribution of $W(t)$, and it was applied in many papers such as Di Crescenzo et al. [3] and Boxma et al. [1]. To derive the c.d.f. of $Y(t)$, we now outline the derivation of the exact distribution of total “on” time $W(t)$ before t in the following subsection.

2.1. Distribution of total “on” time

Consider the same alternating renewal process $(U_1, V_1, U_2, V_2, \dots)$ defined in Section 1. First, we construct a compound renewal process $C(t)$ from the above alternating renewal process. Then, $W(t)$ can be related to the first exit time of $C(t)$ hitting a certain boundary, whose distribution is easier to derive. Define $C(t)$ as follows (see Figure 2)

$$C(t) = \sum_{m=0}^{M(t)} V_m, \quad (2.6)$$

where $M(t)$ is a counting process which gives the total number of complete “up” intervals that their sum is less than or equal to t . More precisely,

$$M(t) = \max\{m \geq 0 : \sum_{j=0}^m U_j \leq t\}. \quad (2.7)$$

Since $U_0 \equiv 0$, $U_1 > t$ implies that $M(t) = 0$. For $m = 0, 1, \dots$, the probability that there are exactly m complete “up” intervals is

$$P\{M(t) = m\} = P\left\{\sum_{j=0}^m U_j \leq t < \sum_{j=0}^{m+1} U_j\right\} = F_U^{(m)}(t) - F_U^{(m+1)}(t), \quad (2.8)$$

where $F_U^{(m)}(t)$ is the m -fold convolution of $F_U(t)$. Here, $F_U^{(0)}(t) = 1$ for $0 \leq t < \infty$. For $m \geq 1$,

$$F_U^{(m)}(t) = \int_0^t f_U(s) F_U^{(m-1)}(t-s) ds. \quad (2.9)$$

Conditioning on $M(t)$, we obtain the c.d.f. of $C(t)$ as

$$K_C(x; t) := P\{C(t) \leq x\} = \sum_{m=0}^{\infty} (F_U^{(m)}(t) - F_U^{(m+1)}(t)) F_V^{(m)}(x). \quad (2.10)$$

Note that $C(t)$ has an atom at 0, i.e.,

$$P\{C(t) = 0\} = 1 - F_U(t). \quad (2.11)$$

Moreover, we note that $\{C(w), w > 0\}$ is a non-decreasing process, and

$$\{W(t) > w\} = \{C(w) < t - w\}$$

as it is illustrated in Figure 2. Therefore, for $0 \leq w < t$,

$$P\{W(t) > w\} = K_C(t - w; w). \quad (2.12)$$

Next, we derive the distribution function of $W(t)$ in the following proposition.

Proposition 2.1. *The distribution function of $W(t)$ is*

$$F_W(w; t) = F_U(w) - \sum_{m=1}^{\infty} (F_U^{(m)}(w) - F_U^{(m+1)}(w)) F_V^{(m)}(t - w), \quad (2.13)$$

for $0 \leq w < t$, and

$$F_W(w; t) = 1, \quad \text{for } w \geq t. \quad (2.14)$$

Proof. The proof of proposition 2.1 is immediately seen by applying the Eq. (2.10), (2.11), and (2.12). □

In order to develop the distribution of $Y(t)$, we also need the density function of $W(t)$ whenever it exists. Notice from the above proposition that the distribution function of $W(t)$ has a jump at $w = t$ and the amount of jump is $P(W(t) = t) = 1 - F_U(t)$. Differentiating $F_W(w; t)$, we obtain

the defective density $f_W(w; t)$ of $W(t)$. For $w \in (0, t)$,

$$\begin{aligned}
 f_W(w; t) &= f_U(w) - \frac{d}{dw} \sum_{m=1}^{\infty} (F_U^{(m)}(w) - F_U^{(m+1)}(w)) F_V^{(m)}(t-w) \\
 &= f_U(w) + \sum_{m=1}^{\infty} (f_U^{(m+1)}(w) - f_U^{(m)}(w)) F_V^{(m)}(t-w) \\
 &\quad + \sum_{m=1}^{\infty} (F_U^{(m)}(w) - F_U^{(m+1)}(w)) f_V^{(m)}(t-w).
 \end{aligned} \tag{2.15}$$

One can see that the defective density $f_W(w, t)$ is always positive by observing the following equation

$$\begin{aligned}
 &f_U(w) + \sum_{m=1}^{\infty} (f_U^{(m+1)}(w) - f_U^{(m)}(w)) F_V^{(m)}(t-w) \\
 &= \sum_{m=0}^{\infty} (F_V^{(m)}(t-w) - F_V^{(m+1)}(t-w)) f_U^{(m+1)}(w).
 \end{aligned} \tag{2.16}$$

Since the distribution of $W(t)$ is derived now, we can easily compute all the moments of $W(t)$ as we state in the following proposition 2.2.

Proposition 2.2. *The k -th moment of $W(t)$ is*

$$E\{(W(t))^k\} = k \int_0^t u^{k-1} K_C(t-u; u) du \tag{2.17}$$

Proof.

$$E\{(W(t))^k\} = t^k(1 - F_U(t)) + \int_0^t w^k f_W(w; t) dw.$$

Expressing $w^k = k \int_0^w u^{k-1} du$, and changing the order of integrals we get

$$\begin{aligned}
 E\{(W(t))^k\} &= t^k(1 - F_U(t)) + k \int_0^t u^{k-1} \int_u^t f_W(w; t) dw du \\
 &= t^k(1 - F_U(t)) + k \int_0^t u^{k-1} (P(W(t) > u) - P(W(t) \geq t)) du \\
 &= t^k(1 - F_U(t)) - t^k(1 - F_U(t)) + k \int_0^t u^{k-1} K_C(t-u; u) du
 \end{aligned} \tag{2.18}$$

□

Now, we are in a position to derive the exact distribution of $Y(t)$ and its moments in the following subsection.

2.2. The distribution function and moments of $Y(t)$

In order to find the exact distribution of $Y(t)$, let us first state the distribution of $Y_i(t)$, $i = 1, 2$. Since $N_i(t)$ and $X_n^{(i)}$ are independent for $i = 1, 2$, and $n = 1, 2, \dots$, conditioning on $N_i(t)$ yields the distribution function of $Y_i(t)$ as

$$H_i(y; t) = e^{-\lambda_i t} + \sum_{m=1}^{\infty} p(m; \lambda_i t) F_i^{(m)}(y), \quad (2.19)$$

for $y \geq 0$, where

$$p(m; \lambda_i t) = e^{-\lambda_i t} \frac{(\lambda_i t)^m}{m!}$$

is the probability mass function of $\text{Poisson}(\lambda_i t)$. The defective density function of $Y_i(t)$ is

$$h_i(y; t) = \sum_{m=1}^{\infty} p(m; \lambda_i t) f_i^{(m)}(y), \quad \text{for } y > 0. \quad (2.20)$$

$H_i(y; t)$ has a discontinuity at $y = 0$. $H_i(0-; t) = 0$ and $H_i(0; t) = e^{-\lambda_i t}$.

Since $\{U_n\}$, $\{V_n\}$ are independent of $Y_i(t)$, $i = 1, 2$, $W(t)$ is independent of $Y_1(t)$ and $Y_2(t)$.

Therefore, from Eq. (2.5), we have

$$P\{Y(t) \leq y | W(t) = w\} = P\{Y_1(w) \leq y + Y_2(t - w)\}. \quad (2.21)$$

Now, conditioning on $Y_2(t)$ and then using the independence of $Y_1(t)$ and $Y_2(t)$, we obtain

$$P\{Y_1(w) \leq y + Y_2(t - w)\} = e^{-\lambda_2(t-w)} H_1(y; w) + \int_0^{\infty} h_2(u; t - w) H_1(y + u; w) du. \quad (2.22)$$

The exponential term on the right side of the above equation appears since $Y_2(t)$ has an atom at

0. Finally, from the equations (2.21) and (2.22), the distribution function of $Y(t)$ is derived in the following proposition by taking expectation of $P\{Y(t) \leq y|W(t) = w\}$ with respect to the distribution of $W(t)$ and using the fact that $W(t)$ has an atom at t by proposition 2.1.

Proposition 2.3. *The distribution function of $Y(t)$ is given by*

$$H_Y(y; t) = (1 - F_U(t))H_1(y; t) + \int_0^t f_W(w; t) \left[e^{-\lambda_2(t-w)} H_1(y; w) + \int_0^\infty h_2(u; t-w) H_1(y+u; w) du \right] dw \quad (2.23)$$

An interesting observation from the above proposition is that the distribution of $Y(t)$ has an atom at 0. Below, we show that the distribution function $H_Y(y; t)$ of $Y(t)$ has a discontinuity (atom) at $y = 0$. Moreover, we find the defective density function of $Y(t)$. Since $H_i(y; t) = 0$ for all $y < 0, i = 1, 2$, we can write $H_Y(y; t)$ more explicitly as

$$H_Y(y; t) = \mathbf{1}_{\{y < 0\}} \left(\int_0^t f_W(w; t) \int_{-y}^\infty h_2(u; t-w) H_1(y+u; w) dudw \right) + \mathbf{1}_{\{y \geq 0\}} \left((1 - F_U(t))H_1(y; t) + \int_0^t f_W(w; t) \left[e^{-\lambda_2(t-w)} H_1(y; w) + \int_0^\infty h_2(u; t-w) H_1(y+u; w) du \right] dw \right). \quad (2.24)$$

From Eq. (2.24), we obtain

$$H_Y(0-; t) = \int_0^t f_W(w; t) \int_0^\infty h_2(u; t-w) H_1(u; w) dw. \quad (2.25)$$

On the other hand, using the Eq. (2.24) and Eq. (2.25) and the fact that $H_1(y; t)$ has an atom at 0 of amount $e^{-\lambda_1 t}$, we write

$$H_Y(0; t) = (1 - F_U(t))e^{-\lambda_1 t} + \int_0^t f_W(w; t) e^{-\lambda_2(t-w) - \lambda_1 w} dw + H_Y(0-; t). \quad (2.26)$$

Thus, $H_Y(y; t)$ has a discontinuity at $y = 0$ of size

$$(1 - F_U(t))e^{-\lambda_1 t} + \int_0^t f_W(w; t)e^{-\lambda_2(t-w) - \lambda_1 w} dw. \quad (2.27)$$

Differentiating $H_Y(y; t)$ and using the fact that $H_i(y; t)$, $i = 1, 2$, has an atom at 0, we obtain the defective density of $Y(t)$ as

$$\begin{aligned} h_Y(y; t) = & \mathbf{1}_{\{y < 0\}} \left(\int_0^t f_W(w; t) \left[e^{-\lambda_1 w} h_2(-y; t-w) + \int_{-y}^{\infty} h_2(u; t-w) h_1(y+u; w) du \right] dw \right) \\ & + \mathbf{1}_{\{y \geq 0\}} \left((1 - F_U(t)) h_1(y; t) + \int_0^t f_W(w; t) \left[e^{-\lambda_2(t-w)} h_1(y; w) \right. \right. \\ & \left. \left. + \int_0^{\infty} h_2(u; t-w) h_1(y+u; w) du \right] dw \right). \end{aligned} \quad (2.28)$$

Next, we derive the moments of $Y(t)$ in the following proposition.

Proposition 2.4. *The k -th moment of $Y(t)$ is*

$$E\{(Y(t))^k\} = \sum_{j=0}^k (-1)^j \binom{k}{j} \int_0^t f_W(w; t) \mu_{k-j}^{(1)}(w) \mu_j^{(2)}(t-w) dw \quad (2.29)$$

Proof. Since $W(t)$, $Y_1(t)$ and $Y_2(t)$ are independent, we have

$$\begin{aligned} E\{(Y(t))^k | W(t) = w\} &= E\{(Y_1(w) - Y_2(t-w))^k\} \\ &= \sum_{j=0}^k (-1)^j \binom{k}{j} E\{(Y_1(w))^{k-j}\} E\{(Y_2(t-w))^j\} \end{aligned} \quad (2.30)$$

For $i = 1, 2$, denote the moment generating function (m.g.f) of $X_1^{(i)}$ and $Y_i(t)$ by $M_X^{(i)}(s)$ and $M_i(s; t)$, respectively. It is not hard to see

$$M_i(s; t) = e^{-\lambda_i t (1 - M_X^{(i)}(s))} \quad (2.31)$$

Therefore, assuming $M_X^{(i)}$ is known, the moments of $Y_i(t)$, denoted by $\mu_l^{(i)}(t) = E\{(Y_i(t))^l\}$, $l =$

$0, 1, 2, \dots$, can be found through Eq. (2.31). Finally, using the equality $E\{(Y(t))^k\} = E\{E\{(Y(t))^k|W(t)\}\}$ we complete the proof of proposition 2.4. \square

We remark that both the distribution function and the defective density function of $Y(t)$ can be computed numerically via some numerical integration method. Moreover, all moments of $Y(t)$ can also be computed from proposition 2.4. We illustrate this in Section 4 through an example.

3 Laplace Transforms

In this section, we derive the double Laplace transform of $W(t)$ and of $Y(t)$. Cohen [2] derived the Laplace Stieltjes transform of $W(t)I_{\{t \in I^+\}}$. Here, we extend the result to the entire time line.

Denote the Laplace transforms of $U_1, V_1, X_1^{(1)}$ and $X_1^{(2)}$ by

$$\begin{aligned}\phi_U(s) &:= \int_0^\infty e^{-st} f_U(t) dt, & \phi_V(s) &:= \int_0^\infty e^{-st} f_V(t) dt, \\ \phi_1(s) &:= \int_0^\infty e^{-st} f_1(t) dt, & \phi_2(s) &:= \int_0^\infty e^{-st} f_2(t) dt.\end{aligned}\tag{3.1}$$

respectively. Let

$$\phi_W(s, \rho) = \int_0^\infty e^{-st} E\{e^{-\rho W(t)}\} dt\tag{3.2}$$

be the double Laplace transform of $W(t)$. Substituting Eq. (2.15) and Eq. (2.16) gives

$$\begin{aligned}\phi_W(s, \rho) &= \sum_{n=0}^\infty \int_{t=0}^\infty e^{-st} \int_{v=0-}^t e^{-\rho(t-v)} \left(F_U^{(n)}(t-v) - F_U^{(n+1)}(t-v) \right) f_V^{(n)}(v) dv dt \\ &\quad + \sum_{n=0}^\infty \int_{t=0}^\infty e^{-st} \int_{u=0-}^t e^{-\rho u} \left(F_V^{(n)}(t-u) - F_V^{(n+1)}(t-u) \right) f_U^{(n+1)}(u) du dt.\end{aligned}\tag{3.3}$$

For $n = 0$, the first renewal cycle covers t . We have

$$\int_0^\infty e^{-st} E\left\{e^{-\rho W(t)} I_{\{t \in I^+\}}\right\} dt = \int_0^\infty e^{-st} e^{-\rho t} (1 - F_U(t)) dt = \frac{1 - \phi_U(s + \rho)}{s + \rho}\tag{3.4}$$

and

$$\begin{aligned} \int_0^\infty e^{-st} E \left\{ e^{-\rho W(t)} I_{\{t \in I_1^-\}} \right\} dt &= \int_0^\infty e^{-st} \int_0^t e^{-\rho u} (1 - F_V(t-u)) du dt \\ &= \frac{1}{s} (1 - \phi_V(s)) \phi_U(s + \rho). \end{aligned} \quad (3.5)$$

Recall that the Laplace transform of the n -fold convolution of a positive function equals the n -th power of the Laplace transform of the function. Therefore, the two parts on the right side of Eq. (3.3) are simplified by separating the case $n = 0$ and changing the order of integrations. Some calculation yields

$$\begin{aligned} &\sum_{n=0}^\infty \int_{t=0}^\infty e^{-st} \int_{v=0-}^t e^{-\rho(t-v)} \left[F_U^{(n)}(t-v) - F_U^{(n+1)}(t-v) \right] dF_V^{(n)}(v) dt \\ &= \frac{1 - \phi_U(s + \rho)}{s + \rho} + \sum_{n=1}^\infty \int_{v=0}^\infty \int_{t=v}^\infty e^{-st} e^{-\rho(t-v)} \left[F_U^{(n)}(t-v) - F_U^{(n+1)}(t-v) \right] dt dF_V^{(n)}(v) \\ &= \frac{1}{s + \rho} \cdot \frac{1 - \phi_U(s + \rho)}{1 - \phi_U(s + \rho) \phi_V(s)}. \end{aligned} \quad (3.6)$$

Similarly,

$$\begin{aligned} &\sum_{n=0}^\infty \int_{t=0}^\infty e^{-st} \int_{u=0-}^t e^{-\rho u} \left[F_V^{(n)}(t-u) - F_V^{(n+1)}(t-u) \right] dF_U^{(n+1)}(u) dt \\ &= \frac{1}{s} (1 - \phi_V(s)) \phi_U(s + \rho) + \sum_{n=1}^\infty \int_{u=0}^\infty e^{-(s+\rho)u} \int_{t=0}^\infty e^{-st} \left[F_V^{(n)}(t) - F_V^{(n+1)}(t) \right] dt dF_U^{(n+1)}(u) \\ &= \frac{\phi_U(s + \rho)}{s} \cdot \frac{1 - \phi_V(s)}{1 - \phi_V(s) \phi_U(s + \rho)}. \end{aligned} \quad (3.7)$$

Thus, $\phi_W(s, \rho)$ is the sum of (3.6) and (3.7) as given in proposition 3.1 below. Now, let us find the double Laplace transform of $Y(t)$ defined as

$$\phi_Y(s, \rho) := \int_0^\infty e^{-st} E \left\{ e^{-\rho Y(t)} \right\} dt.$$

The Laplace transform of $Y(t)$ is

$$E \{ e^{-\rho Y(t)} \} = E \left\{ e^{-\rho (Y_1(W(t)) - Y_2(t - W(t)))} \right\}.$$

The Laplace transform of $Y_i(t)$ is

$$E \{ e^{-\rho Y_i(t)} \} = e^{-\lambda_i t (1 - \phi_i(\rho))}, \quad i = 1, 2. \quad (3.8)$$

Therefore, for $0 < w \leq t$,

$$\begin{aligned} E \{ e^{-\rho Y(t)} | W(t) = w \} &= E \{ e^{-\rho Y_1(w)} \} E \{ e^{\rho Y_2(t-w)} \} \\ &= \exp \left\{ -\lambda_1 w (1 - \phi_1(\rho)) - \lambda_2 (t - w) (1 - \phi_2(-\rho)) \right\} \\ &= \exp \left\{ -w \left(\lambda_1 (1 - \phi_1(\rho)) - \lambda_2 (1 - \phi_2(-\rho)) \right) - \lambda_2 t \left(1 - \phi_2(-\rho) \right) \right\}. \end{aligned} \quad (3.9)$$

Let $a := \lambda_1 (1 - \phi_1(\rho)) - \lambda_2 (1 - \phi_2(-\rho))$ and $b := \lambda_2 (1 - \phi_2(-\rho))$. Taking expectation of $E \{ e^{-\rho Y(t)} | W(t) = w \}$ with respect to the distribution of $W(t)$, we obtain the Laplace transform of $Y(t)$ as $e^{-bt} E \{ -aW(t) \}$. Therefore, the double Laplace transform of $Y(t)$ is

$$\phi_Y(s, \rho) = \int_0^\infty e^{-st} E \{ e^{-\rho Y(t)} \} dt = \int_0^\infty e^{-(s+b)t} E \{ e^{-aW(t)} \} dt.$$

From the definition (3.2), it follows that

$$\phi_Y(s, \rho) = \phi_W \left(s + \lambda_2 (1 - \phi_2(-\rho)), \lambda_1 (1 - \phi_1(\rho)) - \lambda_2 (1 - \phi_2(-\rho)) \right). \quad (3.10)$$

We summarize these results in the following proposition.

Proposition 3.1. *The double Laplace transforms of $W(t)$ and $Y(t)$ are*

$$\phi_W(s, \rho) = \frac{s + \phi_U(s + \rho) (\rho - (s + \rho) \phi_V(s))}{s(s + \rho) (1 - \phi_U(s + \rho) \phi_V(s))} \quad (3.11)$$

and

$$\phi_Y(s, \rho) = \phi_W\left(s + \zeta_2(-\rho), \zeta_1(\rho) - \zeta_2(-\rho)\right), \quad (3.12)$$

where

$$\zeta_i(u) = \lambda_i(1 - \phi_i(u)), i = 1, 2.$$

4 Numerical Results

In this section, we present the formula for distribution function $H_Y(y; t)$ of $Y(t)$ when length of the “up” periods $\{U_n\}$ and the “down” periods $\{V_n\}$ are exponentially distributed with rate parameters θ_u and θ_v respectively, and $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$ follow exponential distributions with rate parameters θ_1 and θ_2 respectively. The random variables $U_n, V_n, X_n^{(1)}$, and $X_n^{(2)}$ are mutually independent for $n = 1, 2, \dots$. Let $g(\cdot; m, \theta)$ and $G(\cdot; m, \theta)$ be the p.d.f. and the c.d.f. of a gamma random variable respectively, where m and θ represent the shape and the rate parameter of gamma random variable respectively. Let us denote the c.d.f. of Poisson(λ) distribution by $\mathcal{P}(\cdot; \lambda)$. Following Eq. (2.15), we obtain for $0 \leq w < t$,

$$\begin{aligned} f_W(w; t) &= g(w; 1, \theta_u) + \sum_{m=1}^{\infty} \{g(w; m+1, \theta_u) - g(w; m, \theta_u)\} G(t-w; m, \theta_v) \\ &\quad + \sum_{m=1}^{\infty} \{G(w; m, \theta_u) - G(w; m+1, \theta_u)\} g(t-w; m, \theta_v) \\ &= \theta_u e^{-\theta_u w} + \theta_u \sum_{m=1}^{\infty} \{p(m; \theta_u w) - p(m-1; \theta_u w)\} (1 - \mathcal{P}(m-1; (t-w)\theta_v)) \\ &\quad + \theta_v \sum_{m=1}^{\infty} p(m; \theta_u w) p(m-1; (t-w)\theta_v) \\ &= e^{-\theta_u w - \theta_v(t-w)} + \theta_u \sum_{m=1}^{\infty} p(m; \theta_u w) p(m; \theta_v(t-w)) + \theta_v \sum_{m=1}^{\infty} p(m; \theta_u w) p(m-1; (t-w)\theta_v) \\ &= e^{-\theta_u w - \theta_v(t-w)} + \sum_{m=1}^{\infty} p(m; \theta_u w) \{ \theta_u p(m; \theta_v(t-w)) + \theta_v p(m-1; (t-w)\theta_v) \} \quad (4.1) \end{aligned}$$

and $P(W(t) = t) = e^{-\theta_u t}$. The defective density function of $Y_2(t)$ is given by

$$h_2(y; t) = \sum_{m=1}^{\infty} p(m; \lambda_2 t) g(y; m, \theta_2) = \theta_2 \sum_{m=1}^{\infty} p(m; \lambda_2 t) p(m-1; \theta_2 y). \quad (4.2)$$

for $y > 0$, and $P(Y_2(t) = 0) = e^{-\lambda_2 t}$. The distribution function of $Y_1(t)$ is given by

$$H_1(y; t) = e^{-\lambda_1 t} + \sum_{m=1}^{\infty} p(m; \lambda_1 t) G(y; m, \theta_1) = 1 - \sum_{m=1}^{\infty} p(m; \lambda_1 t) \mathcal{P}(m-1; \theta_1 y). \quad (4.3)$$

Finally, the distribution function of $Y(t)$ is given by Eq. (2.24) where the jump at 0 is

$$\Delta H_Y(0; t) = e^{-(\theta_u + \lambda_1)t} + \int_0^t f_W(w; t) e^{-\lambda_2(t-w) - \lambda_1 w} dw. \quad (4.4)$$

In Figure 3 and 4, we plot the c.d.f of $Y(t)$ for different values of t and the rate parameters. The plots clearly show the fact that the distribution functions of $Y(t)$ are discontinuous at 0.

In addition to deriving the distribution function of $Y(t)$, we also derive the moments of $Y(t)$ and then study the shape of the distribution of $Y(t)$. In Table 1, we present mean $\mu(t)$, standard deviation $\sigma(t)$, measure of skewness $\gamma_1(t)$, and kurtosis $\gamma_2(t)$ of $Y(t)$ for different rate parameters of exponential distribution. Recall that

$$\gamma_1(t) = \frac{E(Y(t) - \mu(t))^3}{\sigma(t)^3} \quad \text{and} \quad \gamma_2(t) = \frac{E(Y(t) - \mu(t))^4}{\sigma(t)^4} - 3.$$

Note that, for our exponential example, we can write the k^{th} moment of $Y_i(t)$, $i = 1, 2$, as

$$\begin{aligned} \mu_k^{(i)}(t) &= E(Y_i(t)^k) = \int_0^{\infty} y^k h_i(y; t) dy \\ &= \sum_{m=1}^{\infty} p(m; \lambda_i t) \int_0^{\infty} y^k p(m-1; \theta_i y) dy \\ &= \frac{k!}{\theta_i^{k+1}} \sum_{m=1}^{\infty} \binom{k+m-1}{k} p(m; \lambda_i t), \end{aligned} \quad (4.5)$$

for $k = 1, 2, \dots$, and $\mu_0^{(i)}(t) = 1$. Finally, the moments of $Y(t)$ are computed using Eq. (2.29) in proposition 2.4.

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REFERENCES

- [1] Boxma, O., Perry, D., and Zacks, S. (2014). A fluid EOQ model of perishable items with intermittent high and low demand rates. *Mathematics of Operations Research* (to appear).
- [2] Cohen, J. W. (1982). *The single server queue*. Amsterdam: Elsevier Science Publishers.
- [3] Di Crescenzo, A., Iuliano, A., Martinucci, B., and Zacks, S. (2013). Generalized telegraph process with random jumps. *Journal of Applied Probability* 50: 450–463.
- [4] Di Crescenzo, A., Martinucci, B., and Zacks, S. (2014). On the geometric Brownian motion with alternating trends. *Mathematical and Statistical Methods for Actuarial Sciences and Finance*. Edited by Perma, A. and Sibillo, M. New York: Springer.
- [5] Di Crescenzo, A. and Zacks, S. (2014). Probability law and flow function of Brownian motion driven by a generalized telegraph process. *Methodology and Computing in Applied Probability* 17 (to appear).
- [6] Zacks, S. (2004). Generalized integrated telegrapher process and the distribution of related stopping times. *Journal of Applied Probability* 41: 497–507.
- [7] Zacks, S. (2012). Distribution of the total time in a mode of an alternating renewal process with applications. *Sequential Analysis* 31: 397–408.

Table 1. The mean, standard deviation, coefficients of skewness and kurtosis of $Y(t)$ for different values of parameters

parameters	time	$\mu(t)$	$\sigma(t)$	$\gamma_1(t)$	$\gamma_2(t)$
$\lambda_1 = \lambda_2 = 0.5, \theta_1 = \theta_2 = 1, \theta_u = \theta_v = 2$	$t = 1$	0.0550	0.9603	0.3961	14.2859
	$t = 3$	0.1213	1.8191	0.1168	3.9472
	$t = 5$	0.1249	2.3552	0.0546	2.3292
$\lambda_1 = \lambda_2 = 0.5, \theta_1 = \theta_2 = 1, \theta_u = \theta_v = 0.5$	$t = 1$	0.0128	0.6524	0.3627	34.9260
	$t = 3$	0.1404	1.6803	0.2252	5.7384
	$t = 5$	0.2914	2.4416	0.1236	2.6357
$\lambda_1 = \lambda_2 = 0.5, \theta_1 = \theta_2 = 2, \theta_u = \theta_v = 0.5$	$t = 1$	0.0064	0.3262	0.3628	34.9279
	$t = 3$	0.0702	0.8417	0.2292	5.7942
	$t = 5$	0.1457	1.2238	0.1254	2.6689
$\lambda_1 = \lambda_2 = 0.5, \theta_1 = \theta_2 = 2, \theta_u = \theta_v = 2$	$t = 1$	0.0275	0.4801	0.3961	14.2866
	$t = 3$	0.0606	0.9105	0.1191	3.9741
	$t = 5$	0.0624	1.1807	0.0560	2.3677

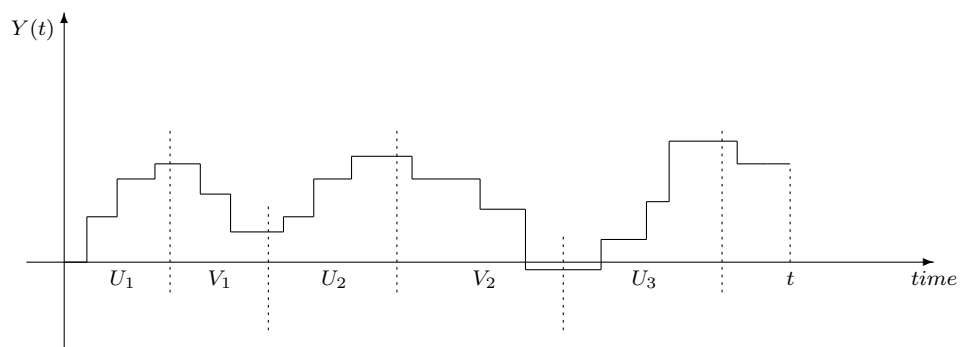


Figure 1. One possible sample path of $Y(t)$ up to time t . $(U_1, V_1, U_2, V_2, \dots)$ is an alternating renewal process. Each increasing piece is a stochastic copy of the compound Poisson process $Y_1(t)$ with positive jumps, while decreasing pieces are stochastic copies of $-Y_2(t)$, a compound Poisson process with negative jumps.

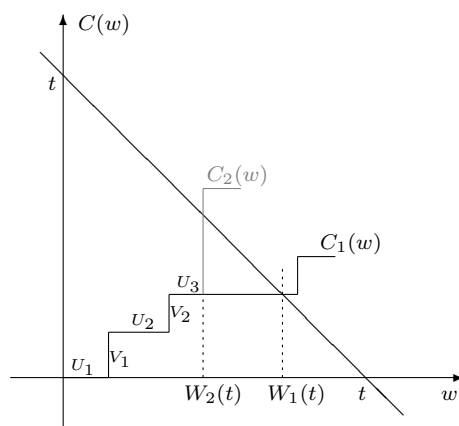


Figure 2. Two possible cases of $C(w)$, namely $C_1(w)$ and $C_2(w)$ crossing the linear boundary $\beta_t(w) = t - w$. Case 1: $C_1(W_1(t)) = t - W_1(t)$. Case 2: $C_2(W_2(t)) > t - W_2(t)$. Notice that in Case 1, $W_1(t) \in I^+$, and in Case 2, $W_2(t) \in I^-$.

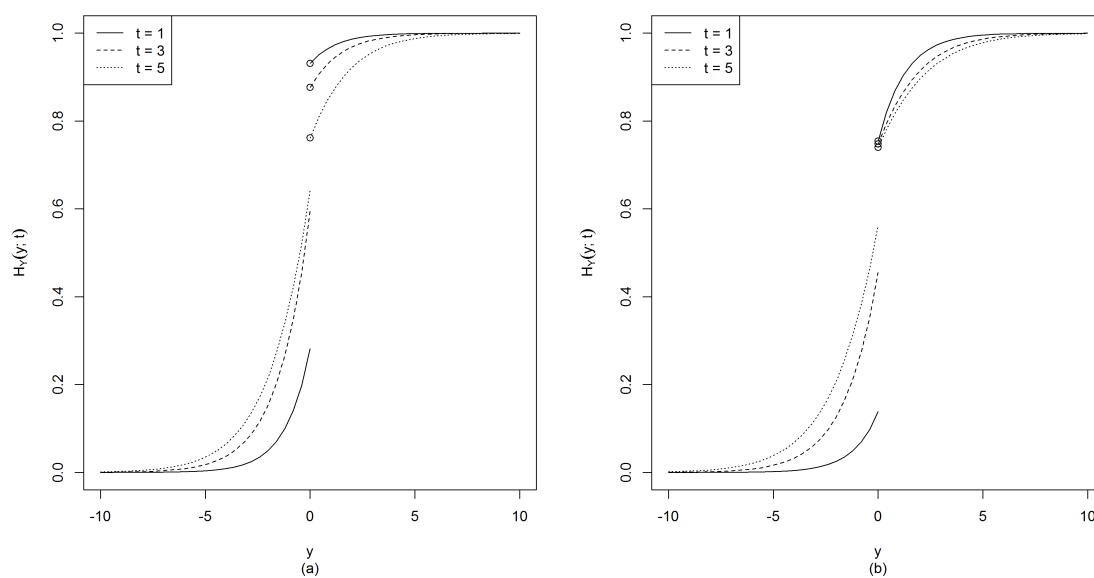


Figure 3. (a) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5, \theta_1 = \theta_2 = 1$, and $\theta_u = \theta_v = 2$. (b) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5, \theta_1 = \theta_2 = 1$, and $\theta_u = \theta_v = 0.5$

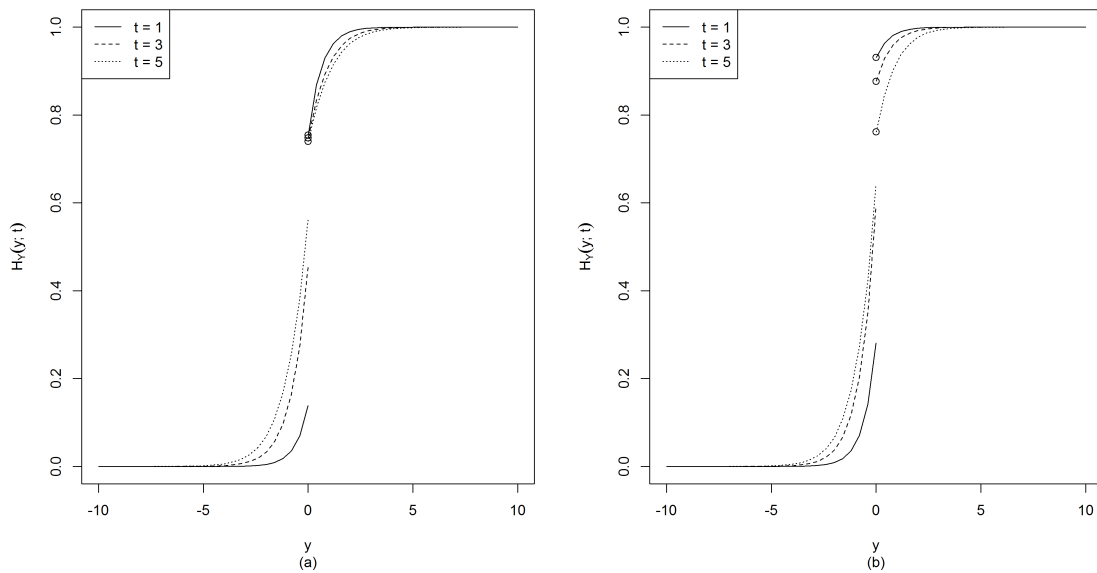


Figure 4. (a) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5$, $\theta_1 = \theta_2 = 2$, and $\theta_u = \theta_v = 0.5$. (b) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5$, $\theta_1 = \theta_2 = 2$, and $\theta_u = \theta_v = 2$