Methodol Comput Appl Probab
DOI 10.1007/s11009-014-9435-2

Exact Calculation of the Distributions of the Stopping Times of Two Types of Truncated SPRT for the Mean of the Exponential Distribution

Shyamal K. De · Shelemyahu Zacks

Received: 1 April 2014 / Revised: 21 November 2014 / Accepted: 19 December 2014
© Springer Science+Business Media New York 2015

Abstract Truncated sequential test procedures are proposed for testing the mean time between failures of a system with exponential life distribution. The exact distributions of the stopping times for the sequential tests are derived by investigating boundary crossing times of homogeneous Poisson processes. The exact formulas for the expected values of the stopping variables and the survival functions are derived. Moreover, the operation characteristic functions are derived exactly for the proposed sequential tests.

Keywords Linear boundaries · Operating characteristic · Poisson process · Sequential testing · Stopping time · Survival function

Mathematics Subject Classification (2010) 60G10 · 60G51 · 62L10 · 62M02

1 Introduction

Consider the problem of testing whether the mean time between failures (MTBF) θ of a system having an exponential life distribution meets a required standard. More specifically, we have to test the hypothesis $H_0 : \theta \leq \theta_0$ against the alternative $H_1 : \theta \geq \theta_1$, where $0 < \theta_0 < \theta_1$, with specified Type I and Type II error probabilities α and β. It is well known that the Wald's Sequential Probability Ratio Test (SPRT) is the most efficient test. If θ_0 and θ_1 are not small, for example if $\theta_0 = 1000$ [hrs] versus $\theta_1 = 2000$ [hrs], the expected length of a sequential test in which we test one system at a time, replacing failed systems by new ones, might be too long (average life length of 1000/24 = 41.7 days).

S. K. De
School of Mathematical Sciences, National Institute of Science Education and Research, Bhubaneswar, Odisha 751005, India
e-mail: sde@niser.ac.in

S. Zacks (✉)
Department of Mathematical Sciences, Binghamton University, 4400 Vestal parkway East, Binghamton, NY 13902, USA
e-mail: shelly@math.binghamton.edu

One can overcome this problem by conducting a sequential test in which s systems operate in parallel. Let X_1 represents the time when the first failure occurs among the s systems, i.e., $X_1 = \min\{T_1, \ldots, T_s\}$, where T_i is the lifetime of the i^{th} system. Clearly, X_1 follows an exponential distribution with mean θ/s since lifetime of each system follows an exponential distribution with mean θ. After the first failure at time X_1, the failed system is replaced immediately by a new one and s systems are put into test again. After time X_1, let X_2 be the additional time needed to observe the next failure from these s systems. In other words, $X_1 + X_2$ is the first time when one of these s systems fails. Thus, $X_1, X_2, \ldots$ are the observed inter-failure times having exponential distribution with MTBF of θ/s. For instance, if $s = 100$ and $\theta = 1000$ [hrs], a failure will be observed on the average every 10 [hrs]. Moreover, if we count the number of renewals (replacing s failed systems), we obtain a homogeneous Poisson process $\{N(t),\ t \geq 0\}$ with intensity $\lambda = \frac{s}{\theta}$ (see Kao 1997).

The OC and ASN functions of SPRT for testing the parameter of an exponential distribution were investigated before by Kemperman (1963), Dvoretzkt et al. (1953), and Ghosh (1970). In this article, our main objective is to derive the distributions of the stopping times, and as a result to obtain exact formulas for the OC and ASN functions.

The operating characteristics of the Wald SPRT can be well approximated (see Siegmund 1985; Woofroofe 1982). In the present paper, we show how one can compute these operating characteristics exactly based on the distributions of the crossing times of the Poisson process $\{N(t), t \geq 0\}$ with truncated SPRT boundaries. If $X_1, X_2, \ldots$ are the exponential interfailure times and $T_n = \sum_{i=1}^{n} X_i$, the Wald's SPRT stops when either:

$$n \ln\left(\frac{\theta_1}{\theta_0}\right) - T_n\left(\frac{\theta_1 - \theta_0}{\theta_0\theta_1}\right) \geq \ln\left(\frac{1-\beta}{\alpha}\right) \tag{1}$$

or

$$n \ln\left(\frac{\theta_1}{\theta_0}\right) - T_n\left(\frac{\theta_1 - \theta_0}{\theta_0\theta_1}\right) \leq -\ln\left(\frac{1-\alpha}{\beta}\right). \tag{2}$$

Replacing n by $N(t)$ and T_n by t, we obtain corresponding boundaries for $\{N(t), t \geq 0\}$, namely,

$$N(t) \geq a_2 + bt \quad \text{and} \quad N(t) \leq -a_1 + bt, \tag{3}$$

where $a_1 = \ln\left(\frac{1-\alpha}{\beta}\right)\left(\ln\left(\frac{\theta_1}{\theta_0}\right)\right)^{-1}$, $a_2 = \ln\left(\frac{1-\beta}{\alpha}\right)\left(\ln\left(\frac{\theta_1}{\theta_0}\right)\right)^{-1}$, and $b = \left(\frac{\theta_1-\theta_0}{\theta_0\theta_1}\right)\left(\ln\left(\frac{\theta_1}{\theta_0}\right)\right)^{-1}$. Making the transformations $t' = bt$ and $\mu = \lambda/b$, where $\lambda = 1/\theta$, we reduce the problem to the problem of finding the distribution of the crossing times of the Poisson process $\left\{N(t'), t' \geq 0\right\}$ with intensity $\mu = \lambda/b$ and the boundaries:

$$B_L(t') = -a_1 + t' \quad \text{and} \quad B_U(t') = a_2 + t'. \tag{4}$$

In the present paper, we consider two types of truncated SPRT.

In Type I SPRT, the lower boundary is $B_L^*(t) = -k_1 + t$ and the upper boundary is $B_U^*(t) = k_2$ where k_1 and k_2 are positive integers (see Fig. 1). In this Type of sequential testing, testing stops when $N(t) = -k_1 + t$ or when $N(t) = k_2$, whichever comes first.

In Type II SPRT, we have two parallel boundaries $B_L(t) = -k_1 + t$ and $B_U(t) = k_2 + t$ and a frequency truncation point m where k_1, k_2, and m are positive integers (see Fig. 5). In this scenario, testing stops as soon as $N(t) \leq B_L(t)$ or $N(t) \geq B_U(t)$ or $N(t) \geq m$.

In Sections 2 and 3, we develop the distributions of the corresponding stopping times, the expected values of the stopping times, and the operating characteristic curve $\pi(\mu) = P_\mu\,\{\text{accepting } H_0\}$. The OC and ASN functions are plotted in terms of μ which is given as $\mu = \frac{1}{\theta b}$ where b is given below the inequalities in Section 3. In Section 4, we discuss the performance of Type I and Type II designs. The "R" functions used for computations of

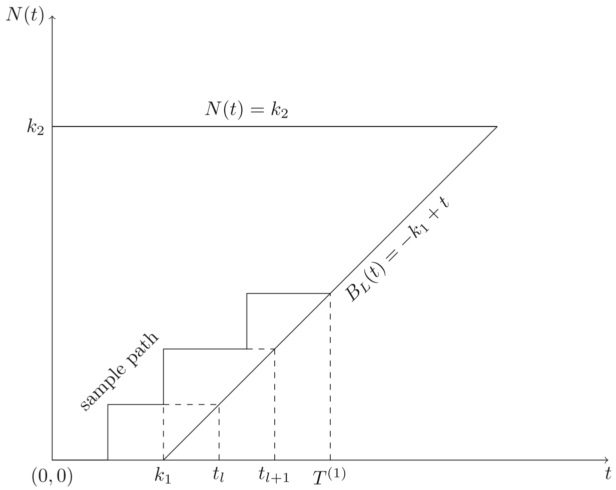

Fig. 1 Sequential testing with linear lower boundary

OC functions, survival functions, and expected values of stopping times for both Type I and Type II designs can be obtained in the website: http://www.math.binghamton.edu/sde/.

2 Type I Sequential Testing

We devote this section to develop the exact distribution and expected value of the stopping variable for Type I sequential testing. We also derive the operating characteristic and survival function of the stopping time.

2.1 The Stopping Boundaries

Without loss of generality, we develop the theory for the stopping variable with the lower boundary $B_L(t) = -k_1 + t$, and frequency truncation at $N(t) = k_2$, as shown in Fig. 1. The stopping time is

$$T^{(1)} = \inf\{t \geq 0 : N(t) = -k_1 + t \text{ or } N(t) = k_2\}. \tag{5}$$

If the Poisson process $N(t)$ reaches the lower boundary, i.e., if $N(T^{(1)}) = -k_1 + T^{(1)}$, we accept H_0. If $N(T^{(1)}) = k_2$, we reject H_0. Clearly, $T^{(1)} \leq k_1 + k_2 - 1$ with probability one.

2.2 The Operating Characteristic of $T^{(1)}$

An important characterization of any sequential test procedure is its operating characteristic (OC) function $\pi^{(1)}(\mu) = P_\mu(\text{accepting } H_0)$. To compute the OC function, we define

$$\psi_\mu^{(1)}(j; k_1, k_2) = P_\mu\left\{N\left(T^{(1)}\right) = j\right\} \text{ for } j = 0, \ldots, k_2 - 1.$$

The function $\psi_\mu(j; k_1, k_2)$ represents the probability that the Poisson process crosses the lower boundary at level j. The values of t at which $N(t)$ can cross the lower boundary are $t_l = k_1 + l$, where $l = 0, \ldots, k_2 - 1$. Let $p(.; \mu)$ and $P(.; \mu)$ denote the probability mass function and cumulative distribution function of the Poisson distribution with mean μ. The values of $\psi_\mu(j; k_1, k_2)$ can be found recursively according to the equations

$$\psi_\mu^{(1)}(j; k_1, k_2) = \begin{cases} e^{-\mu k_1} & \text{if } j = 0, \\ p(j; \mu t_j) - \sum_{i=0}^{j-1} \psi_\mu^{(1)}(i; k_1, k_2) p(j-i; \mu(j-i)) & \text{if } j = 1, 2, \ldots, k_2 - 1. \end{cases} \tag{6}$$

The first term inside the summation represents the probability that $N(t)$ crosses the lower boundary at height j and the second term represents the probability that $N(t)$ reaches height j starting from height i in time $t_j - t_i$. Since the second term of Eq. 6 is a convolution, one can derive an exact formula for $\psi_\mu^{(1)}(j; k_1, k_2)$ by using generating functions (Zacks 1991; 2009). Following Zacks (1991), we obtain

$$\psi_\mu^{(1)}(j; k_1, k_2) = \sum_{i=0}^{j} q_i(\mu) p(j - i; \mu(j - i + k_1)), \tag{7}$$

where

$$q_i(\mu) = \begin{cases} 1, & \text{if } i = 0, \\ (-1)^i \det\{A_i(\mu)\} & \text{if } i \geq 1, \end{cases}$$

and $\det\{A_i(\mu)\}$ is the determinant of the $i \times i$ matrix

$$A_i(\mu) = \begin{bmatrix} p(1; \mu) & p(2; 2\mu) & \cdots & p(i; i\mu) \\ 1 & p(1; \mu) & \cdots & p(i-1; (i-1)\mu) \\ 0 & 1 & \cdots & p(i-2; (i-2)\mu) \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & p(1; \mu) \end{bmatrix}.$$

Notice that $A_i(\mu)$ is an $i \times i$ submatrix of the infinite dimensional semi-circulant matrix $A(\mu)$. Applying equation (7), the OC function is obtained as

$$\begin{aligned} \pi^{(1)}(\mu) &= \sum_{j=0}^{k_2-1} \psi_\mu^{(1)}(j; k_1, k_2) \\ &= \sum_{j=0}^{k_2-1} \sum_{i=0}^{j} q_i(\mu) p(j - i; \mu(j - i + k_1)) \\ &= \sum_{i=0}^{k_2-1} q_i(\mu) \sum_{j=0}^{k_2-1-i} p\left(j; \mu(k_1 + j)\right). \end{aligned} \tag{8}$$

In Fig. 2, we present the OC function $\pi^{(1)}(\mu)$, for the case of $k_1 = 3$ and $k_2 = 40$.

2.3 Exact Distribution of $T^{(1)}$

In the present subsection, we derive the exact formula for the survival function $S_\mu^{(1)}(t; k_1, k_2) = P_\mu\{T^{(1)} > t\}$ and the expected stopping time $E\left(T^{(1)}\right)$. Simplifying the notation, we will write $S^{(1)}(t)$ to denote this survival function. Since $\{N(t), t \geq 0\}$ is a non-decreasing process, $S^{(1)}(t) = P\left(k_2 - 1; \mu t\right)$ if $t \leq k_1$ (see Fig. 1). In order to compute $S^{(1)}(t)$ for $t > k_1$, we introduce the following defective probability function

$$g_\mu^{(1)}(j; t) = P_\mu\left\{N(t) = j,\ T^{(1)} > t\right\}. \tag{9}$$

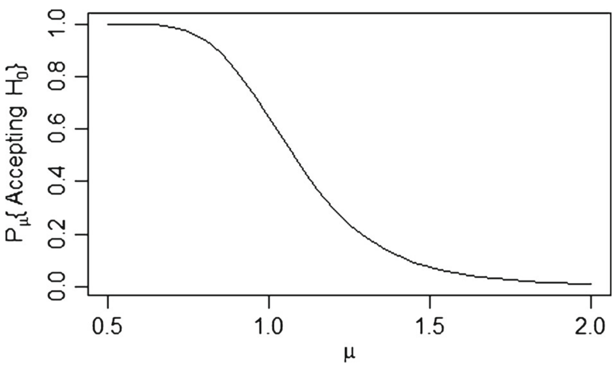

Fig. 2 The OC function for Type I sequential testing with $k_1 = 3$ and $k_2 = 40$

This defective probability function can be computed at time points t_l, for $l = 1, 2, \ldots, k_2 - 1$, as

$$g_\mu^{(1)}(j; t_l) = p(j; \mu t_l) - \sum_{i=0}^{l-1} \psi_\mu^{(1)}(i; k_1, k_2) p(j - i; \mu(l - i)). \tag{10}$$

The first term inside the summation represents the probability that $N(t)$ crosses the lower boundary for the last time (before attaining height j) at height i. The second term inside the summation represents the probability that $N(t)$ reaches height j from height i in time $t_l - t_i$. For other time points and $j = l + 1, \ldots, k_2 - 1$,

$$g_\mu^{(1)}(j; t) = \begin{cases} p(j; \mu t), & \text{if } 0 < t \le t_0, \\ \sum_{i=l+1}^{j} g_\mu^{(1)}(i; t_l) p(j - i; \ \mu(t - t_l)) & \text{if } t_l < t < t_{l+1}, \end{cases} \tag{11}$$

where $l = 0, \ldots, k_2 - 1$.

The survival function can be computed by adding the defective probability functions at different heights. Thus, for $l = 0, 1, \ldots, k_2 - 1$,

$$\begin{aligned} P_\mu \left\{ T^{(1)} > t_l \right\} &= \sum_{j=l+1}^{k_2-1} g_\mu^{(1)}(j; t_l) \\ &= P(k_2 - 1; \mu t_l) - P(l; \mu t_l) \\ &\quad - \sum_{j=l+1}^{k_2-1} \sum_{i=0}^{l-1} \psi_\mu^{(1)}(i; k_1, k_2) p(j - i; \mu(l - i)) \\ &= P(k_2 - 1; \mu t_l) - P(l; \mu t_l) \\ &\quad - \sum_{i=0}^{l-1} \psi_\mu^{(1)}(i; k_1, k_2) \sum_{j=l+1}^{k_2-1} p(j - i; \mu(l - i)) \\ &= P(k_2 - 1; \mu t_l) - P(l; \mu t_l) \\ &\quad - \sum_{i=0}^{l-1} \psi_\mu^{(1)}(i; k_1, k_2) [P(k_2 - 1 - i; \ \mu(l - i)) - P(l - i; \ \mu(l - i))] \end{aligned} \tag{12}$$

In Table 1, we present a few values of the survival probability $S^{(1)}(t)$.

Table 1 Values of $S^{(1)}(t)$ for $k_1 = 3, k_2 = 7, \mu = 1$

t	l	$S^{(1)}(t; 3, 7, 1)$
1		0.9999
2		0.9955
3	0	0.9665
4	1	0.7846
5	2	0.6072
6	3	0.4092
7	4	0.2244
8	5	0.0825

If $t_l < t < t_{l+1}, l = 0, 1, \ldots, k_2 - 1$, then

$$P_\mu \left\{ T^{(1)} > t \right\} = \sum_{j=l+1}^{k_2-1} g_\mu^{(1)}(j; t_l) P\left(k_2 - 1 - j; \mu(t - t_l)\right). \tag{13}$$

Notice that if $t \le k_1 = t_0$, $P_\mu \left\{ T^{(1)} > t \right\} = P(k_2 - 1; \mu t)$.

2.4 Expected Value of $T^{(1)}$

Following (12) and (13), we obtain

$$\begin{aligned} E_\mu\{T^{(1)}\} &= \int_0^{k_1+k_2-1} P_\mu \left\{ T^{(1)} > t \right\} dt \\ &= \int_0^{k_1} P(k_2 - 1; \mu t)\, dt + \sum_{l=0}^{k_2-1} \sum_{j=l+1}^{k_2-1} g_\mu^{(1)}(j; t_l) \int_{t_l}^{t_{l+1}} P(k_2 - 1 - j; \mu(t - t_l))\, dt \\ &= \frac{1}{\mu} \sum_{i=0}^{k_2-1} (1 - P(i; k_1\mu)) + \frac{1}{\mu} \sum_{l=0}^{k_2-1} \sum_{j=l+1}^{k_2-1} g_\mu^{(1)}(j; t_l) \sum_{i=0}^{k_2-1-j} (1 - P(i; \mu)). \end{aligned} \tag{14}$$

In Figs. 3 and 4, we present $E_\mu \left\{ T^{(1)} \right\}$ and $S_\mu^{(1)}(t; k_1, k_2)$ respectively, for $k_1 = 3, k_2 = 40$, and $\mu = 1, 1.5, 2$.

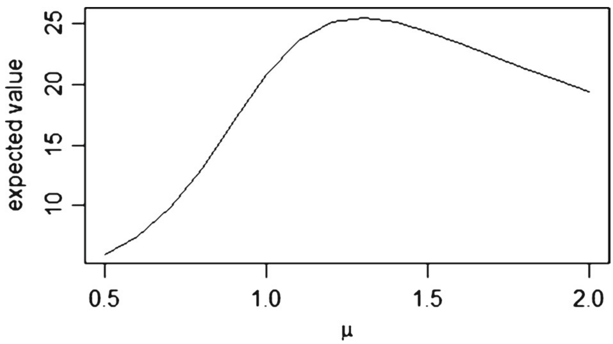

Fig. 3 Expected value of $T^{(1)}$ for $k_1 = 3$ and $k_2 = 40$

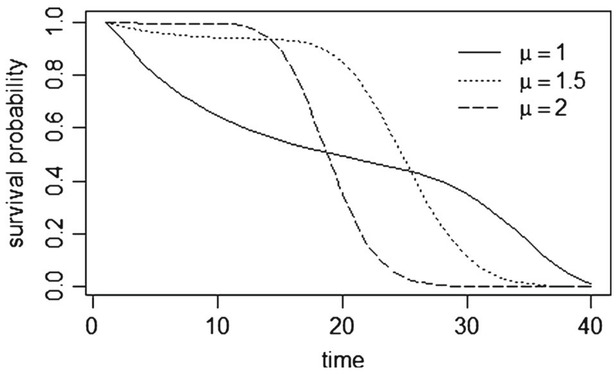

Fig. 4 The survival functions of $T^{(1)}$ for $k_1 = 3$ and $k_2 = 40$

In the following section, we present another type of truncated sequential testing where sampling stops when the Poisson process $N(t)$ crosses some parallel linear boundaries.

3 Type II Sequential Testing

In Type II testing, the boundaries are two parallel lines $B_L(t) = -k_1 + t$ and $B_U(t) = k_2 + t$, and the truncation at $N(t) = m$. In Fig. 5, we present the continuation region. We define the following stopping times

$$\begin{aligned} T_L &= \inf\{t > 0 : N(t) = -k_1 + t\} \\ T_U &= \inf\{t > 0 : N(t) \geq k_2 + t\} \\ T^* &= \inf\{t > 0 : N(t) = m\} \\ T^{(2)} &= \min\left\{T_L,\ T_U,\ T^*\right\} \end{aligned} \tag{15}$$

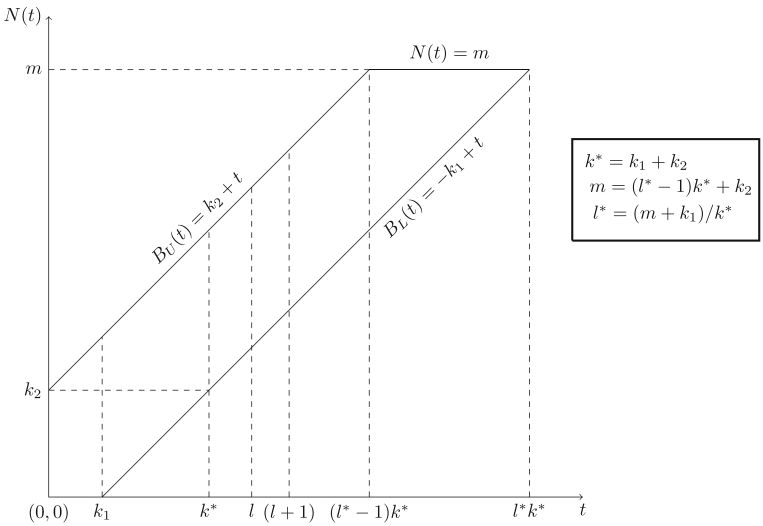

Fig. 5 The continuation region of Type II sequential testing

Notice that $P_\mu\{T_L \geq k_1\} = 1$ and $P_\mu\{T_L = k_1\} = e^{-\mu k_1}$. Moreover, T_L is a discrete random variable assuming values in $\{t_l : l = 0, 1, \ldots, l^* k^* - 1\}$ where $k^* = k_1 + k_2$, $l^* = (m + k_1)/k^*$, and m is the truncation level. In order to compute the distribution of $T^{(2)}$, the probability $\pi^{(2)}(\mu) = P_\mu\left\{T^{(2)} = T_L\right\}$ and $E_\mu\left\{T^{(2)}\right\}$, we introduce the defective probability function

$$g_\mu^{(2)}(j;t) = P_\mu\left\{N(t) = j,\ T^{(2)} > t\right\}. \tag{16}$$

3.1 Computation of $g_\mu^{(2)}(j;t)$

The values of $g_\mu^{(2)}(j;t)$ are positive only for values of j inside the continuation region, i.e., between the two parallel boundaries and smaller than m. Moreover, $0 < t < l^* k^* = k_1 + m$ (see Fig. 6).

First, for $0 < t \leq k_1$, if $0 \leq j \leq k_2$, then clearly $g_\mu^{(2)}(j;t) = p(j;\mu t)$. However, if $k_2 < j < k_2 + t$, we first compute the values of $g_\mu^{(2)}(j;t)$ at $l = 1, \ldots, k_1$ and $k_2 < j \leq k_2 + l - 1$,

$$g_\mu^{(2)}(j;l) = p(j;\mu l) - \sum_{i=k_2+1}^{j} g_0(j-i, l-i+k_2;\mu)\, p(i;\mu(i-k_2)), \tag{17}$$

where

$$g_0(j;\ t,\ \mu) = P_\mu\{N(t) = j,\ \ N(s) < s \quad \forall\ 0 < s \leq t\}$$

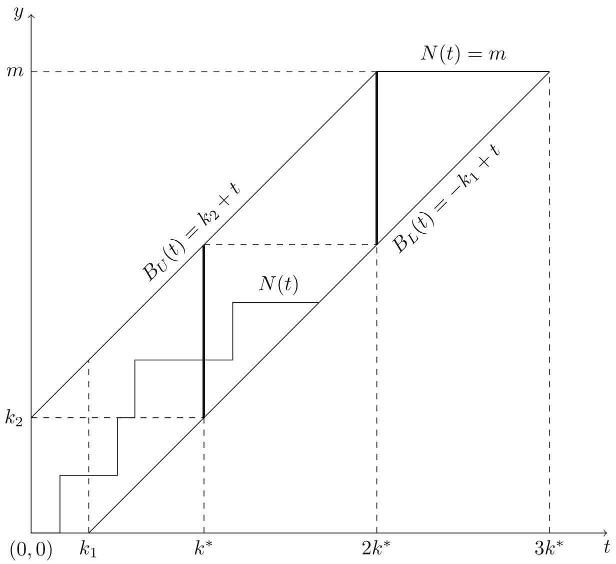

Fig. 6 A three-block structure in Type II sequential testing

represents the probability that the Poisson process is at height j in time t without crossing the line $N(t) = t$. The Eq. 17 is obtained by conditioning on the last entrance of $N(t)$ to the region below the line $N(t) = t$. As shown in Zacks (1991), we obtain a recursive equation whose solution is

$$g_0(j, j+k; \mu) = \sum_{i=0}^{j} q_i(\mu) p\left(j-i; \mu(j-i+k)\right). \tag{18}$$

Following these we can compute, for $l < t < l+1, l \leq k_1$,

$$g_\mu^{(2)}(j; t) = \sum_{i=0}^{j} g_\mu^{(2)}(i; l) p\left(j-i; \mu(t-l)\right). \tag{19}$$

After we compute the values of $g_\mu^{(2)}(j; k_1)$, we proceed recursively for each t_l, $l = 0, \ldots, (l^* - 1)k^*$ and $l + 2 \leq j \leq k^* + l - 1$ according to the formula

$$g_\mu^{(2)}(j; t_{l+1}) = \sum_{i=l+1}^{\min\{j,\ (k^*+l-2)\}} g_\mu^{(2)}(i; t_l) p\left(j-i; \mu\right). \tag{20}$$

Finally, for $(l^* - 1)k^* + 1 \leq l \leq l^* k^* - 1$, we compute the defective probability function in Eq. 20 only for $l + 2 \leq j \leq m - 1$.

3.2 Computation of the OC Function

In order to compute the OC function, we have to compute for each t_l the probability

$$\psi_\mu^{(2)}(l; k_1, k_2, m) = P_\mu \left\{T_L = t_l,\ T_L < T_U\right\}. \tag{21}$$

For this purpose, we partition the continuation region into l^* blocks. Without loss of generality, assume that $m = l^* k^* - k_1$. In this case, each block is supported by a time interval of length k^*. Thus, the n^{th} block, $n = 1, \ldots, l^*$, is

$$B_n = \left\{(t, y) : (n-1)k^* \leq t \leq nk^*,\ \max\left\{0, -k_1 + t\right\} \leq y \leq \min\left\{m,\ k_2 + t\right\}\right\} \tag{22}$$

A sample path of the Poisson process $\{N(t), t \geq 0\}$ which crosses the lower boundary $B_L(t)$, for $k_1 \leq t \leq k^*$, cannot have crossed the upper boundary $B_U(t)$ before. Hence,

$$\psi_\mu^{(2)}(l; k_1, k_2, m) = \psi_\mu^{(1)}(l; k_1, m) \quad \text{if } l \leq k^* - k_1. \tag{23}$$

Suppose that $\left\{T^{(2)} > k^*\right\}$ and $N(k^*) = j$ for some $j = k^* - k_1 + 1, \ldots, 2k^* - k_1 - 1$. Conditioning on $N(k^*) = j$, the sample path might cross $B_L(t)$ at levels $\{j, j+1, \ldots, 2k^* - k_1 - 1\}$ without crossing $B_U(t)$. Since $N(t)$ is strongly Markovian, we can use the function $\psi_\mu^{(1)}\left(.;\ k_1^{(2)},\ k_2^{(2)}\right)$ with $k_1^{(2)} = j - k_2$ and $k_2^{(2)} = 2k^* - k_1 - j$ to obtain

$$\pi_{j,\ l}^{(2)} = P_\mu \left\{N(T_L) = l, T_L < T_U \mid N(k^*) = j,\ T^{(2)} > k^*\right\}.$$

Introduce a vector of dimension k^*, $\boldsymbol{\pi}^{(2)}(j)$, of these conditional probabilities

$$\pi^{(2)}(j)^T = \left(0,\ 0, \ldots, 0,\ \pi_{j,\ j}^{(2)},\ \pi_{j,\ j+1}^{(2)}, \ldots, \pi_{j,\ 2k^*-1}^{(2)}\right)$$

with $(j-1)$ zeros at the beginning, and for $l = j + k_1, \ldots, 2k^* - 1$,

$$\boldsymbol{\pi}_{j,\ l}^{(2)} = \psi_\mu^{(1)}\left(l - k^*;\ j - k_2,\ 2k^* - k_1 - j\right) \tag{24}$$

Notice that $\psi_{\mu}^{(1)}$ was computed with the parameters k_1 and k_2 in Section 2. Thus, $\psi_{\mu}^{(1)}\left(.;\, k_1^{(2)},\, k_2^{(2)}\right)$ is the function $\psi_{\mu}^{(1)}(.;)$ with the parameters $k_1^{(2)} = j - k_2$ and $k_2^{(2)} = 2k^* - k_1 - j$. Finally, the vector of probabilities for the second block is

$$\boldsymbol{\pi}^{(2)} = \sum_{j=k^*-k_1+1}^{2k^*-k_1-1} g_{\mu}^{(2)}\left(j;\; k^*\right)\boldsymbol{\pi}^{(2)}(j). \tag{25}$$

The l^{th} component of $\boldsymbol{\pi}^{(2)}$, $l = k^* + 1, \ldots, 2k^*$, is $P_{\mu}\{T_L = t_l,\, T_L < T_U\} = \pi^{(2)}(l)$. Similarly, we can compute the vector $\boldsymbol{\pi}^{(n)}$ for the n^{th} block, i.e.,

$$\boldsymbol{\pi}^{(n)} = \sum_{j=(n-1)k^*-k_1+1}^{nk^*-k_1-1} g_{\mu}^{(2)}\left(j;\; (n-1)k^*\right)\boldsymbol{\pi}^{(n)}(j). \tag{26}$$

In Fig. 7, we present the OC function $\boldsymbol{\pi}^{(2)}(\mu)$ for Type II testing with $k_1 = 3, k_2 = 7$, and $m = 40$.

3.3 The Survival Function and Expectation of $T^{(2)}$

In Section 3.1, we presented formulae of the defective probability function function $g_{\mu}^{(2)}(j;t)$. Using this function, we can compute

$$P_{\mu}\left\{T^{(2)} > t\right\} = \sum_{j=\lfloor -k_1+t\rfloor^+ +1}^{\min\left\{\lfloor k_2+t\rfloor^*,\; m-1\right\}} g_{\mu}^{(2)}(j;t), \tag{27}$$

where $\lfloor a \rfloor$ = the largest integer that is smaller or equal to a and

$$\lfloor k_2 + t\rfloor^* = \begin{cases} \lfloor k_2 + t\rfloor + 1 & \text{if } t = 1, 2, 3, \ldots \\ \lfloor k_2 + t\rfloor & \text{if } t \text{ is not a positive integer.} \end{cases}$$

If $k_1 \le l < t < l + 1 \le (l^* - 1)k^*$, then

$$P_{\mu}\left\{T^{(2)} > t\right\} = \sum_{j=l-k_1+1}^{l+k_2-1} g_{\mu}^{(2)}(j;l)P\left(k_2 + l - 1 - j;\; \mu(t - l)\right). \tag{28}$$

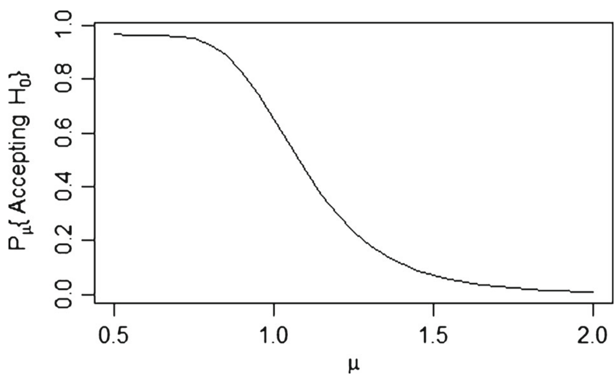

Fig. 7 The OC curve for Type II sequential testing with $k_1 = 3$, $k_2 = 7$, and $m = 40$

If $l < t < l+1$ and $l \le k_1 - 1$, then

$$P_\mu\left\{T^{(2)} > t\right\} = \sum_{j=1}^{l+k_2-1} g_\mu^{(2)}(j;l)P\left(k_2+l-1-j;\ \mu(t-l)\right). \tag{29}$$

and if $(l^*-1)k^* < t < l^*k^*$, then

$$P_\mu\left\{T^{(2)} > t\right\} = \sum_{j=-k_1+\lfloor t\rfloor+1}^{m-1} g_\mu^{(2)}\left(j;\ \lfloor t\rfloor\right) P\left(m-1-j;\ \mu\left(t-\lfloor t\rfloor\right)\right). \tag{30}$$

To find the expected value of $T^{(2)}$, we can use the formula

$$\begin{aligned} E_\mu\left\{T^{(2)}\right\} &= \int_0^{l^*k^*} P_\mu\left\{T^{(2)} > t\right\} \\ &= \int_0^1 P(k_2;\mu t)dt + \left(\sum_{l=2}^{k_1} + \sum_{l=k_1+1}^{l^*k^*-2}\right)\int_{t_{l-1}}^{t_l} P_\mu\left\{T^{(2)} > t\right\}dt. \end{aligned}$$

Notice that

$$\int_0^1 P(k_2;\mu t)dt = \frac{1}{\mu}\sum_{i=0}^{k_2}\left(1-P(i;\mu)\right).$$

Thus, using Eq. 28, we obtain

$$\begin{aligned} E_\mu\left\{T^{(2)}\right\} = \frac{1}{\mu}\Bigg[&\sum_{i=0}^{k_2}\left(1-P(i;\mu)\right) + \sum_{l=2}^{k_1}\sum_{j=1}^{l+k_2-1} g_\mu^{(2)}(j;l)\sum_{i=0}^{l+k_2-1-j}\left(1-P(i;\mu)\right) \\ &+ \sum_{l=k_1+1}^{(l^*-1)k^*}\sum_{j=l-k_1+1}^{l+k_2-1} g_\mu^{(2)}(j;l)\sum_{i=0}^{l+k_2-1-j}\left(1-P(i;\mu)\right) \\ &+ \sum_{l=(l^*-1)k^*+1}^{l^*k^*-2}\sum_{j=l-k_1+1}^{m-1} g_\mu^{(2)}(j;l)\sum_{i=0}^{m-1-j}\left(1-P(i;\mu)\right)\Bigg] \end{aligned} \tag{31}$$

The values of $E_\mu\left\{T^{(2)}\right\}$ are plotted in Fig. 8, for the case of $k_1 = 3$, $k_2 = 7$, $m = 40$. The survival functions of $T^{(2)}$ are plotted in Fig. 9.

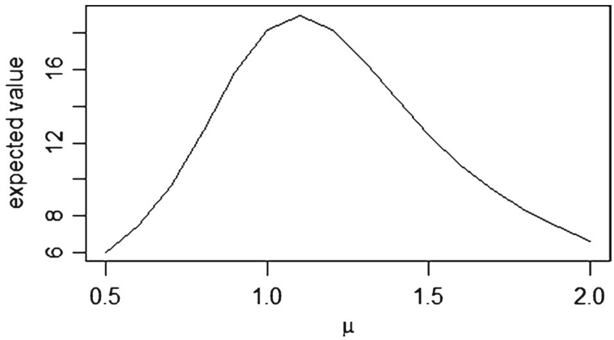

Fig. 8 Expected value of $T^{(2)}$ for $k_1 = 3$, $k_2 = 7$, and $m = 40$

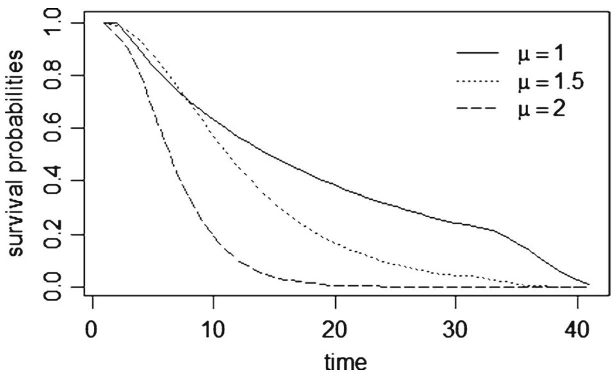

Fig. 9 The survival functions of $T^{(2)}$ for $k_1 = 3$, $k_2 = 7$, and $m = 40$

4 Discussion

In this section, we present a numerical comparison between the characteristics of the two types of truncated sequential designs discussed in Sections 2 and 3. For this purpose, we refer to the numerical example in the introduction where we considered the case of $\theta_0 = 1000$ [hrs] and $\theta_1 = 2000$ [hrs]. These correspond to $\lambda_0 = 1/1000 = 10^{-3}$ and $\lambda_1 = 1/2000 = 5 \times 10^{-4}$. For the error probabilities $\alpha = \beta = 0.05$, SPRT has two parallel lines with slope $b = 0.0007213$ and intercepts -4.25 and 4.25. Making the transformation to $\mu = \lambda/b$, we obtain $\mu_0 = 0.6932$ and $\mu_1 = 1.3863$. In Table 2, we compare the OC functions $\pi^{(i)}(\mu)$ and the expected values $E_\mu\left\{T^{(i)}\right\}$ for $i = 1, 2$ with $k_1 = 3$ and $k_2 = 40$ for the Type I sequential testing and $k_1 = 3$, $k_2 = 7$, and $m = 40$ for the Type II sequential testing.

Table 2 illustrates that the OC functions, i.e., probabilities of crossing the lower boundaries (accepting H_0), for the Type I and Type II sequential designs are nearly the same. On the other hand, we note that the Type II design requires less expected sample size to finish the experiement than that of the Type I design. Moreover, we remark that both $T^{(1)}$ and $T^{(2)}$ are proper in the sense that $P_\mu\left\{T^{(1)} < \infty\right\} = 1$ and $P_\mu\left\{T^{(2)} < \infty\right\} = 1$.

In general, Wald's SPRT can only provide approximations to the OC function and the expected value of the stopping time. In this work, the OC function and the expected value of the stopping time are computed *exactly*. Notice that the parameters $k_1 = 3$, $k_2 = 7$, and $m = 40$ yield, as shown in Table 2, an actual value of type I error probability $\alpha \cong 0.042$ and type II error probability $\beta \cong 0.112$. With the aid of the "R" programs in the Appendix, we obtain error probabilities $\alpha \cong 0.041$ and $\beta \cong 0.068$ for parameters $k_1 = 3$, $k_2 = 9$, and

Table 2 Comparing operating characteristics of Type I and Type II sequential designs

θ	μ	$\pi^{(1)}(\mu)$	$\pi^{(2)}(\mu)$	$E_\mu\left\{T^{(1)}\right\}$	$E_\mu\left\{T^{(2)}\right\}$
2000	0.7	0.987	0.958	9.764	9.657
1386.4	1.0	0.642	0.650	20.782	18.198
1000	1.4	0.116	0.112	25.134	14.358
866.5	1.6	0.046	0.044	23.372	10.745
603.2	2	0.008	0.008	19.345	6.614

$m = 40$. Thus, the exact formulas provide algorithms that facilitates the design of efficient sequential tests.

To elaborate the importance of exact calculation of the OC and ASN functions, we emphasize that they are required for obtaining the exact type I and type II error probabilities and the expected sample size corresponding to the test parameters k_1, k_2, and m. The OC functions can be used to plan the experiment by deriving appropriate stopping boundaries. For Type I sequential testing of $H_0 : \mu \geq \mu_0$ against $H_1 : \mu < \mu_1$, if one wishes to control the probabilities of Type I and type II errors at specified levels α and β respectively, the parameters (k_1, k_2) can be chosen by numerically solving $1 - \pi^{(1)}(\mu_0, k_1, k_2) = \alpha$ and $\pi^{(1)}(\mu_1, k_1, k_2) = \beta$. In the absence of exact solutions of these equations, one can choose (k_1, k_2) such that $|1 - \pi^{(1)}(\mu_0, k_1, k_2) - \alpha|$ and $|\pi^{(1)}(\mu_1, k_1, k_2) - \beta|$ are minimum. For Type II sequential testing, one may choose the boundary parameters (k_1, k_2, m) such that weighted expected sample size $w_0 E_{\mu_0}\left\{T^{(2)}\right\} + w_1 E_{\mu_1}\left\{T^{(2)}\right\}$ is minimized under the constraints $1 - \pi^{(2)}(\mu_0, k_1, k_2, m) \leq \alpha$ and $\pi^{(2)}(\mu_1, k_1, k_2, m) \leq \beta$ for some known weights w_0 and w_1 for H_0 and H_1 respectively such that $w_0 + w_1 = 1$. Moreover, to plan a reliability experiment (such as deciding about s, the number of systems to put under test), one needs to have estimates of the expected length of the trials. We mention this problem in Section 1. This article provides *exact* formulas for the expected sample sizes which facilitate the planning of an experiment. In addition, we provide an algorithm for calculating the exact distribution of the stopping times $T^{(1)}$ and $T^{(2)}$.

Notice that the CDF of $T^{(i)}$, for $i = 1, 2$, is continuous on intervals (t_l, t_{l+1}), $l = 0, 1, \ldots$, and have jumps of size $\psi^{(i)}$ on the integers t_l.

Finally, we remark that a generalization to the case of compound Poisson process with parallel boundaries was given by Xu (2012).

References

Dvoretzkt A, Kiefer J, Wolfowitz J (1953) Sequential decision problems for processes with continuous time parameter. testing hypotheses. Ann Math Stat 24(2):254–264

Ghosh BK (1970) Sequential tests of statistical hypotheses. Addison-Wesley, Reading

Kao EPC (1997) An introduction to stochastic processes. Duxbury, New York

Kemperman JHB (1963) A Weiner-Hopf type method for a general random walk with a two-sided boundary. Ann Math Stat 34(4):1168–1193

Siegmund D (1985) Sequential analysis: tests and confidence intervals. Springer, New York

Woodroofe M (1982) Nonlinear renewal theory in sequential analysis. SIAM monograph

Xu Y (2012) First exit times of compound Poisson processes with parallel boundaries. Sequential Analysis: Tests and Confidence Intervals 31(2):135–144

Zacks S (1991) Distributions of stopping times for Poisson processes with linear boundaries. Communications in Statistics. Stochastic Models 7:233–242

Zacks S (2009) Stage-wise adaptive designs. Wiley, New York