Frames of twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ and shift-invariant spaces on the Heisenberg group

Abstract: A well known result on translates of a function φ in $L^2(\mathbb{R})$ states that the collection $\{\tau_k\varphi : k \in \mathbb{Z}\}$ forms an orthonormal system in $L^2(\mathbb{R})$ iff $p_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+k)|^2 = 1$ *a.e.* $\xi \in \mathbb{T}$. Similarly in the literature there are interesting characterizations of Bessel sequence, frame, Riesz basis of a system of translates in $L^2(\mathbb{R})$ in terms of Fourier transform.

In this talk, we aim to study frames in twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ and shift-invariant spaces on the Heisenberg group $\mathbb{H}^n$. First we shall define a twisted shift-invariant space $V^t(\varphi)$ in $L^2(\mathbb{R}^{2n})$ as the closed linear span of the twisted translates of φ. We shall obtain characterizations of orthonormal system, Bessel sequence, frame and Riesz basis consisting of twisted translates $\{T^t_{(k,l)}\varphi : k,l \in \mathbb{Z}^n\}$ of $\varphi \in L^2(\mathbb{R}^{2n})$ in terms of the kernel K_φ of the Weyl transform of φ. In particular, we shall prove that if $\{T^t_{(k,l)}\varphi : k,l \in \mathbb{Z}^n\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$, then $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$, where $w_\varphi(\xi) = \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2 d\eta,\ \xi \in \mathbb{T}^n$. Unlike the classical case on $\mathbb{R}^n$, it turns out to be a surprising fact that the converse of the above theorem need not be true. The converse is true with an additional condition, which we call "condition C". In fact, we shall see that $\{T^t_{(k,l)}\varphi : k,l \in \mathbb{Z}^n\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$ if and only if $w_\varphi(\xi) = 1$ a.e. $\xi \in \mathbb{T}^n$ and φ satisfies condition C. Next we shall study shift-invariant spaces associated with countably many mutually orthogonal generators $\mathcal{A}$ on the Heisenberg group. We shall conclude the talk by providing a sampling formula on a subspace of a twisted shift-invariant space as an application of our results.